ELECTRIC FLUX DENSITY, GAUSS'S LAW, AND DIVERGENCE

1. Electric Flux Density (D) (كثافة التدفق الكهربائي)

Michael Faraday had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (dielectric material*)* which would occupy the entire volume between the concentric spheres

His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as displacement, displacement flux, or simply electric flux.

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere

$$\Psi = Q$$

Where Ψ (psi) is electric flux in coulombs C

We can obtain more quantitative information by considering an inner sphere of radius *a* and an outer sphere of radius b, with charges of Q and — Q, respectively (Fig. 3.1). The paths of electric flux Ψ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

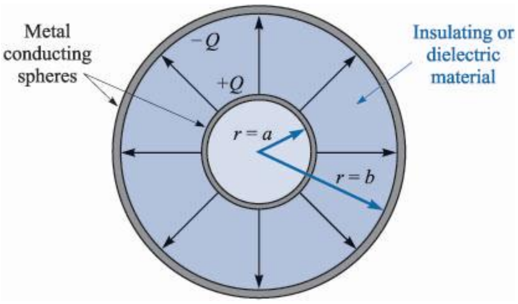


Figure 3.1: The electric flux in the region between a pair of charged concentric sphere

At the surface of the inner sphere, Ψ coulombs of electric flux are produced by the charge Q(=Ψ) coulombs distributed uniformly over a surface having an area of $4\pi a^2$ m^2. The density of the flux at this surface is $\Psi/4\pi a^2$ or $Q/4\pi a^2$ C/m^2, and this is an important new quantity.

Referring again to Fig. 3.1, the electric flux density is in the radial direction and has a value of

$$\vec{D}(\text{at } r = a) = \frac{Q}{4\pi a^2}\mathbf{a}_r \quad \text{(inner sphere)}$$

$$\vec{D}(\text{at } r = b) = \frac{Q}{4\pi b^2}\mathbf{a}_r \quad \text{(outer sphere)}$$

and at a radial distance r, where $a \le r \le b$

$$\vec{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

If we now let the inner sphere become smaller and smaller, while still retaining a charge of Q, it becomes a point charge in the limit, but the electric flux density at a point r meters from the point charge is still given by

$$\vec{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

$$\vec{D} = \varepsilon_o \vec{E} \qquad \text{(in free space)}$$

Example: Determine D at (4, 0, 3) if there is a point charge -5π mC at (4, 0, 0) and a line charge 3π mC/m along the y-axis?

Solution:

$D = D_1 + D_2$

D_1 *duoto point charge*

D_2 *duoto line charge*

$$D_1 = \frac{Q}{4\pi|R_1|^2}\mathbf{a}_{\boldsymbol{R_1}}$$

$R_1 = (4-4)\mathrm{a}_x + (0-0)\mathrm{a}_y + (3-0)\mathrm{a}_z$

$R_1 = 3\mathrm{a}_z$

$|R_1| = 3$

$$\mathbf{a}_{\boldsymbol{R_1}} = \frac{3\mathrm{a}_z}{3} = \mathrm{a}_z$$

$$D_1 = \frac{-5\pi \times 10^{-3}}{4\pi 3^2}\mathrm{a}_z$$

$D_1 = -0.138\ \mathrm{a}_z\ \ mC/m^2$

$$D_2 = \frac{\rho_L}{2\pi}\frac{x\mathrm{a}_x + z\mathrm{a}_z}{(x^2 + z^2)}$$

$$D_2 = \frac{3\pi \times 10^{-3}}{2\pi}\frac{4\mathrm{a}_x + 3\mathrm{a}_z}{(4^2 + 3^2)}$$

$D_2 = 0.24\ \mathrm{a}_x + 0.18\ \mathrm{a}_z\ \ mC/m^2$

$D = D_1 + D_2 = 0.24\ \mathrm{a}_x + 0.042\ \mathrm{a}_z\ \ mC/m^2$

2. Gauss's Law

These generalizations of Faraday's experiment lead to the following statement, which is known as **Gauss's law**:

"The electric flux passing through any closed surface is equal to the total charge enclosed by that surface"

قانون جاوس :"التدفق الكهربي المار خلال اي سطح مغلق يساوي الشحنة الكلية المحتواة بذلك السطح"

$$\Psi = Q_{enclosed}$$

$\Delta\Psi = D_S . \Delta S$

$$\Psi = \oint D_S \; . dS$$

توضع دائرة صغيرة على علامة التكامل لتشير الى ان التكامل مؤدى على سطح مغلق ويسمى هذا السطح ب "سطح جاوس"

- ***The charge enclosed might be several point charges, in which case***

$$Q_{enclosed} = \sum Q_n$$

- ***or a line charge,***

$$Q_{enclosed} = \int \rho_L dL$$

- ***or a surface charge,***

$$Q_{enclosed} = \int_S \rho_S \, ds$$

- ***or a volume charge distribution,***

$$Q_{enclosed} = \int_{vol} \rho_v \, dv$$

The last form is usually used, and we should agree now that it represents any or all of the other forms. With this understanding Gauss's law may be written in terms of the charge distribution as

$$\oint D_S \; . dS = \int_{vol} \rho_v dv$$

3. Applications of Gauss's Law

The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists. Once symmetric charge distribution exists, we construct a mathematical closed surface (known as a *Gaussian surface).* The surface over which Gauss's law is applied must be closed, but it can be made up of several surface elements. Thus the defining conditions of a *special* ***Gaussian surface*** are

- **a- The surface is closed.**
- **b- At each point of the surface D is either normal or tangential to the surface, so that Ds. dS becomes either DsdS or zero, respectively**
- **c- D is sectional constant over that part of the surface where D is normal.**

3.1 Symmetrical Charge Distributions:

3.1.1 Point Charge:

Suppose a point charge Q is located at the origin. To determine D at a point P, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions. Thus, a ***spherical surface*** centered at the origin is the ***Gaussian surface*** in this case and is shown in Figure 3.2.

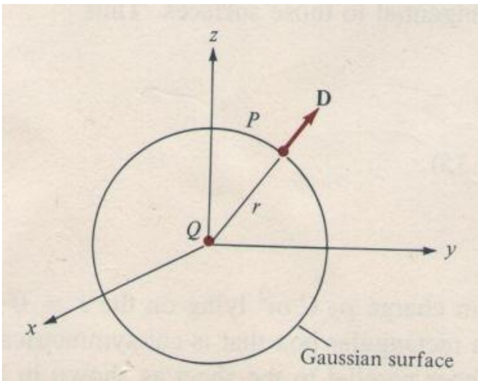


Figure 3.2

$$\Psi = \oint D_S \cdot dS = D_r \oint dS$$

$$\Psi = D_r \int_{\emptyset=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta d\emptyset = 4\pi r^2 D_r$$

$$Q_{enclosed} = Q$$

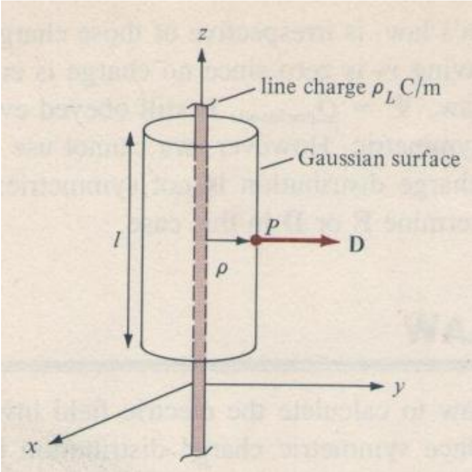
$$\Psi = Q_{enclosed}$$

$$4\pi r^2 D_r = Q$$

$$D = \frac{Q}{4\pi r^2} \mathbf{a}_r$$

3.1.2 Infinite Line Charge

Suppose the infinite line of uniform charge ρ_LC/m lies along the z-axis. To determine D at a point P, we choose a ***cylindrical surface*** containing P to satisfy symmetry condition as shown in Figure 3.3. D is constant on and normal to the cylindrical Gaussian surface; i.e., D $=D_\rho \mathbf{a}_\rho$. If we apply Gauss's law to an arbitrary length L of the line



$$\Psi = \oint D_S \, . dS = D_\rho \oint dS$$

$$\Psi = D_r \int_{z=0}^{l} \int_{\emptyset=0}^{2\pi} \rho d\emptyset dz$$

$$\Psi = 2\pi\rho L D_\rho$$

$$Q_{enclosed} = \int \rho_L dL$$

$$Q_{enclosed} = \rho_L \int_{z=0}^{L} d_z = L\,\rho_L$$

$$\Psi = Q_{enclosed}$$

$$2\pi\rho L D_\rho = L\,\rho_L$$

$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

$$D = \frac{\rho_L}{2\pi\rho}\,\mathbf{a}_\rho$$

3.1.3 Uniformly Charged Sphere

Consider a sphere of radius a with a uniform charge ρ_v C/m3. To determine D everywhere, we construct Gaussian surfaces for cases r < a, and r > a separately. Since the charge has spherical symmetry, it is obvious that a ***spherical surface*** is an appropriate Gaussian surface.

For **r** < ***a***, the total charge enclosed by the spherical surface of radius r, as shown in Figure 3.4 (a), is

$$\Psi = \oint D_S \ .dS = D_r \oint dS = D_r \int_{\emptyset=0}^{2\pi}\int_{\theta=0}^{\pi} r^2 \sin\theta \ d\theta d\emptyset = 4\pi r^2 D_r$$

$$Q_{enclosed} = \int_{vol} \rho_v \ dv \ = \rho_v \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^2 \sin\theta \ dr d\theta d\emptyset = \frac{4}{3}\rho_v \pi \ r^3$$

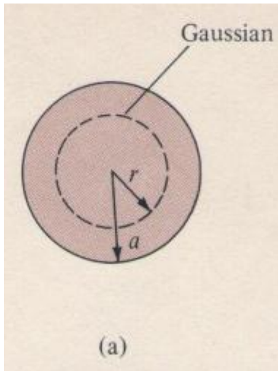


Figure 3.4-a

$$\Psi = Q_{enclosed} \qquad \text{(Gauss's law)}$$

$$4\pi r^2 D_r = \frac{4}{3}\rho_v \pi \ r^3$$

$$\boxed{D = \frac{\rho_v}{3} \ r \ \mathbf{a}_r}$$

For **r** ≥ ***a***, the Gaussian surface is shown in Figure 4.3(b). The charge enclosed by the surface is the entire charge in this case, i.e.,

$$Q_{enclosed} = \int_{vol} \rho_v \ d_v \ = \rho_v \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta \ dr d\theta d\emptyset = \frac{4}{3}\rho_v \pi \ a^3$$

$$\Psi = Q_{enclosed} \qquad \text{(Gauss's law)}$$

$$4\pi r^2 D_r = \frac{4}{3}\rho_v \pi \ a^3$$

$$D_r = \frac{a^3}{3r^2} \ \rho_v$$

$$\boxed{D = \frac{a^3}{3r^2} \ \rho_v \ \mathbf{a}_r} \qquad (r \geq a)$$

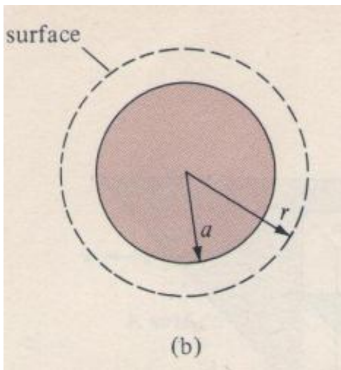
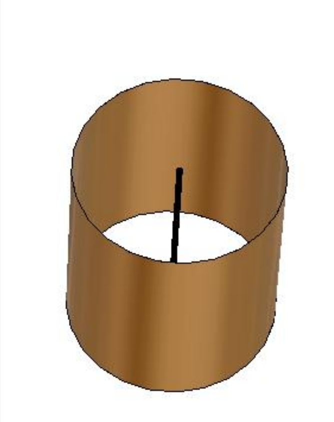


Figure 3.4-b

Example: A uniform line charge of $\rho_L = 3\mu$ C/m lies along the z axis, and a concentric circular cylinder of radius 2 m has $\rho_s = \frac{-1.5}{4\pi}$ μC/m^2. Both distributions are infinite in extent with z. Use Gauss's law to find D in all regions?

Solution:



$1 - \ The\ region\ 0 < \rho < 2$

Using Gaussian surface cylinder ρ

$\Psi = Q_{enclosed}$

$$\Psi = D_\rho \int_{z=0}^{l} \int_{\emptyset=0}^{2\pi} \rho d\emptyset dz = 2\pi\rho L D_\rho$$

$$Q_{enclosed} = \int \rho_L dL = \rho_L \int_{z=0}^{L} d_z = L\ \rho_L$$

$2\pi\rho L D_\rho = L\ \rho_L$

$$D = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho = \frac{3\times10^{-6}}{2\pi\rho}\mathbf{a}_\rho = \frac{0.477}{\rho}\mathbf{a}_\rho \quad \mu\, C/m^2$$

$2 - \ The\ region\ \ 2 < \rho$

$Q_{enclosed} = Q_1 + Q_2$

$Q_1 = L\ \rho_L$

$$Q_2 = \int \rho_s\, dS$$

$$Q_2 = \rho_s \int_0^L \int_0^{2\pi} \rho\, d\emptyset\, dz = 2\times 2\pi \times L \times \rho_s = 4\pi L\rho_s$$

$Q_{enclosed} = Q_1 + Q_2 = L\ \rho_L + 4\pi L\rho_s$

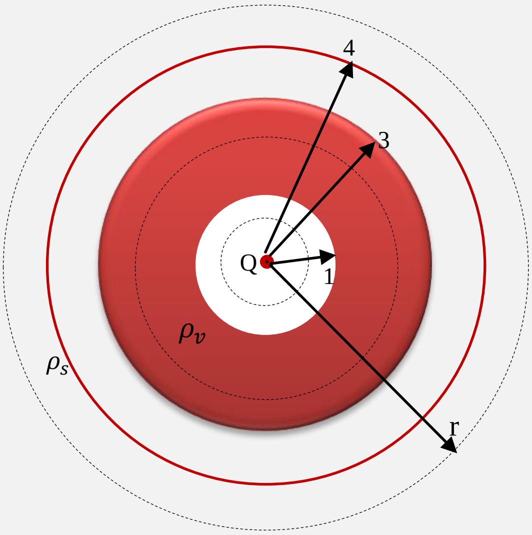
$\Psi = Q_{enclosed}$

$2\pi\rho L D_\rho = L\ \rho_L + 4\pi L\rho_s$

$$D = \frac{\rho_L + 4\pi\rho_s}{2\pi\rho}\ \mathbf{a}_\rho = \frac{0.239}{\rho}\mathbf{a}_\rho \quad \mu\, C/m^2$$

Example: *A point charge* ***Q=2π c*** *at the origin , a* ***volume*** *charge density of* ***4 c/m³*** *at the region* $\mathbf{1 \le r \le 3}$ *and a* ***sheet*** *of charge* ***-6 c/m²*** *has* $\boldsymbol{r = 4}$, *Find* $\boldsymbol{D}$ *at* $\boldsymbol{r = 0.5}$, $\boldsymbol{r = 2}$, $\boldsymbol{r = 5}$?

Solution:



D at $r = 0.5$

D is due to a point charge

$$D = \frac{Q}{4\pi r^2}\,\mathbf{a}_r = \frac{2\pi}{4\pi(0.5)^2}\,\mathbf{a}_r = 2\,\mathbf{a}_r$$

$\boldsymbol{D}$ at $\boldsymbol{r = 2}$

$$\Psi = Q_{enclosed}$$

$$\Psi = 4\pi r^2 D_r$$

$$Q_{enclosed} = Q_1 + Q_2$$

$$Q_2 = \int_{vol} \rho_v\, d_v = 4\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta\, dr d\theta d\emptyset = 4 * 2 * 2\pi * \left[\frac{r^3}{3}\right]_1^2 = \frac{112\pi}{3}$$

$$\therefore 4\pi r^2 D_r = 2\pi + \frac{112\pi}{3}$$

$$D = \frac{39.33}{4(2)^2}\mathbf{a}_r = 2.45\,\mathbf{a}_r$$

$\boldsymbol{D}$ at $\boldsymbol{r = 5}$

$$Q_{enclosed} = Q_1 + Q_2 + Q_3$$

$$Q_2 = 4 * 2 * 2\pi * \left[\frac{r^3}{3}\right]_1^3 = \frac{416\pi}{3}$$

$$Q_3 = \int \rho_s\, dS = -6\int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta\, d\theta d\emptyset = -6 * (4)^2 * 2\pi * 2 = -384\pi$$

$$\therefore 4\pi r^2 D_r = 2\pi + \frac{416\pi}{3} - 384\pi = -243.33\pi$$

$$D_r = \frac{-243.33\pi}{4\pi r^2}\,\mathbf{a}_r = \frac{-243.33}{4(5)^2}\,\mathbf{a}_r = -2.433\,\mathbf{a}_r$$

3.2 Differential Volume Element

We are now going to apply the methods of Gauss's law to a slightly different type of problem, one which does not possess any symmetry at all. At first glance it might seem that our case is hopeless, for without symmetry a **simple Gaussian surface cannot be chosen** such that the normal component of D is constant or zero everywhere on the surface. Without such a surface, the integral cannot be evaluated. There is only one way to circumvent these difficulties, and that is to choose such a very small closed surface that D is *almost* constant over the surface, and the small change in D may be adequately represented by using the first two terms of the Taylor's-series expansion for D. The result will become more nearly correct as the volume enclosed by the Gaussian surface decreases, and we intend eventually to allow this volume to approach zero.

$$Charge\ enclosed\ in\ volume\ \Delta v = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times volume\ \Delta v$$

The expression is an approximation which becomes better as Δv becomes smaller.

Example: Find an approximate value for the total charge enclosed in an incremental volume of 10^{-9} m^3 located at the origin, if $D = e^{-x} \sin y\, a_x - e^{-x} \cos y\, a_y + 2z a_z$ C/m^2.

Solution:

$$\frac{\partial D_x}{\partial x} = e^{-x}\sin y\ , \quad \frac{\partial D_y}{\partial y} = e^{-x}\sin y\ , \quad \frac{\partial D_z}{\partial z} = 2$$

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately $2\Delta v$.If Δv is $10^{-9}m^3$, then we have enclosed about 2nC.

4. <u>**Divergence**</u> (div)

There are two main indicators of the manner in which a vector field changes from point to point throughout space. The first of these is divergence, which will be examined here. It is a scalar and bears a similarity to the derivative of a function. The second is curl.

When the divergence of a vector field is nonzero, that region is said to contain sources or sinks, sources when the divergence is positive, sinks when negative. In static electric fields there is a correspondence between positive divergence, sources, and positive electric charge Q. Electric flux ψ by definition originates on positive charge. Thus, a region which contains positive charges contains the sources of ψ. The divergence of the electric flux density D will be positive in this region. A similar correspondence exists between negative divergence, sinks, and negative electric charge

Divergence of A=div A= $\lim_{\Delta v \to 0} \frac{\oint \mathrm{A}\ . dS}{\Delta v}$

div D=∇.D $= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$ (Cartesian)

div D=∇.D $= \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho D_\rho + \frac{1}{\rho}\frac{\partial D_\emptyset}{\partial \emptyset} + \frac{\partial D_z}{\partial z}$ (cylindrical)

div D=∇.D $= \frac{1}{r^2}\frac{\partial}{\partial r}r^2 D_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\emptyset}{\partial \emptyset}$ (spherical)

5. Maxwell's First Equation (Electrostatics)

We now wish to consolidate the gains of the last two sections and to provide an interpretation of the divergence operation as it relates to electric flux density. The expressions developed there may be written as

div D= $\lim_{\Delta v \to 0} \frac{\oint D_S\ . dS}{\Delta v}$

div D $= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$

div D $= \rho_v$

$\nabla.\mathrm{D} = \rho_v$

This is the first of Maxwell's four equations as they apply to electrostatics and steady magnetic fields, and it states that ***the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there***. This equation is called the ***point form*** *of Gauss's law*. Gauss's law relates the flux leaving any closed surface to the charge enclosed,

Example: in the region $a \le \rho \le b$, $\vec{\mathbf{D}} = \rho_0 \left(\frac{\rho^2 - a^2}{2\rho}\right) \mathbf{a}_{\boldsymbol{\rho}}$,

and for $\rho > b$ $\vec{\mathbf{D}} = \rho_0 \left(\frac{b^2 - a^2}{2\rho}\right) \mathbf{a}_{\boldsymbol{\rho}}$

for $\rho < a$ $\vec{\mathbf{D}} = \mathbf{0}$, find $\rho_v$in all three regions?

Solution:

$1 - \ for\ the\ region\ a \le \rho \le b$

$$D_\rho = \rho_0 \left(\frac{\rho^2 - a^2}{2\rho}\right) \quad , \quad D_\emptyset = 0 \quad , \quad D_z = 0$$

$$\nabla . \mathrm{D} = \rho_v$$

$$\rho_v = \nabla . \mathrm{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho D_\rho + \frac{1}{\rho}\frac{\partial D_\emptyset}{\partial \emptyset} + \frac{\partial D_z}{\partial z}$$

$$\rho_v = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\left(\rho_0\left(\frac{\rho^2 - a^2}{2\rho}\right)\right) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho_0\left(\frac{\rho^2 - a^2}{2}\right) = \frac{\rho_0}{2\rho}\frac{\partial}{\partial \rho}(\rho^2 - a^2) = \frac{\rho_0}{2\rho} \times 2\rho = \rho_0\ C/m^3$$

$2 - \ for\ the\ region\ \ \rho > b$

$$\rho_v = \nabla . \mathrm{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho D_\rho = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\left(\rho_0\left(\frac{b^2 - a^2}{2\rho}\right)\right) == \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho_0\left(\frac{b^2 - a^2}{2}\right) = \frac{\rho_0}{2\rho}\frac{\partial}{\partial \rho}(b^2 - a^2) = 0$$

$3 - \ for\ the\ region\ \ \rho < a$

$$\rho_v = \nabla . \mathrm{D} = \rho_v = \nabla . 0 = 0$$

Example: Let $\overrightarrow{\mathbf{D}} = 5r^2\mathbf{a}_r$ mC/m^2, for $r \leq 0.08$ m and $\overrightarrow{\mathbf{D}} = \frac{0.1}{r^2}\mathbf{a}_r$ mC/m^2, for $r > 0.08$ m, (a) find ρ_v at r =0.06m, (b) find ρ_v at r =0.1m, (c) what surface charge density could be located at r =0.08m to caused D=0 for > 0.08 ?

Solution:

$(\boldsymbol{a})\ \ \rho_v\ at\ r = 0.06$

$$for\ r < 0.08 \qquad \overrightarrow{\mathbf{D}} = 5r^2\mathbf{a}_r\ \text{mC/m}^2$$

$$\rho_v = \nabla.D = \frac{1}{r^2}\frac{\partial}{\partial r}r^2D_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\emptyset}{\partial\emptyset}$$

$$\rho_v = \frac{1}{r^2}\frac{\partial}{\partial r}r^2(5r^2) = \frac{1}{r^2}\frac{\partial}{\partial r}5r^4 = \frac{1}{r^2}\ 20r^3 = 20r \quad mC/m^3$$

$$\rho_{v_{at\ r=0.06}} = 20(0.06) \times 10^{-3} = 1.2\, mC/m^3$$

$(\boldsymbol{b})\ \ \rho_v\ at\ r = 0.1$

$$for\ r > 0.08$$

$$\overrightarrow{\mathbf{D}} = \frac{0.1}{r^2}\mathbf{a}_r\ \ \text{mC/m}^2$$

$$\rho_v = \nabla.D = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\frac{0.1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}0.1 = 0$$

$(\boldsymbol{c})\ \ D = 0 \Rightarrow \Psi = 0 \ \ \Rightarrow Q_{enclosed} = 0$

$$Q_{enclosed} = Q_1 + Q_2$$

$$Q_2 = -Q_1 = -\int \rho_v dv = -\int_0^{2\pi}\int_0^{\pi}\int_0^{0.08} 20r\, r^2\sin\theta\, dr\, d\theta d\emptyset$$

$$Q_2 = -\int_0^{2\pi}\int_0^{\pi}\int_0^{0.08} 20\, r^3\sin\theta\, dr\, d\theta d\emptyset = -\, 20 \times 2\pi \times 2 \times \left[\frac{r^4}{4}\right]_0^{0.08} = -2.57\,\mu C$$

$$Q_2 = \int \rho_s dS = \rho_s\int_0^{2\pi}\int_0^{\pi} r^2\sin\theta\ \ d\theta d\emptyset = (0.08)^2 \times 2 \times 2\pi \times \rho_s$$

$$-2.57 \times 10^{-6} = 0.0804\,\rho_s$$

$$\rho_s = -32\,\mu C/m^2$$

6. The Divergence Theorem

Gauss' law states that the closed surface integral of D. dS is equal to the charge enclosed. If the charge density function ρ_v is known throughout the volume, then the charge enclosed may be obtained from an integration of ρ_v throughout the volume. Thus,

$$\oint D_S \, .dS = \int_{vol} \rho_v dv$$

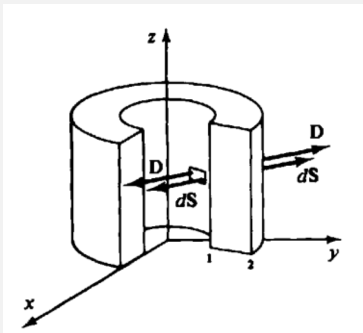
But $\nabla . \mathrm{D} = \rho_v$ and so

$$\oint D_S \, .dS = \int_{vol} (\nabla . \mathrm{D}) dv$$

This is the ***divergence theorem***, also known as *Gauss' divergence theorem*. Of course, the volume v is that which is enclosed by the surface s.

Example: Given that $D = (10\rho^3/4)\, a\rho$, (C/m^2) in cylindrical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by ρ= 1m, ρ= 2m, z = 0 and z = 10

Solution:



The left side of divergence theorem is:

$$\oint D_S \, .dS = \int_0^{10} \int_0^{2\pi} \frac{10\rho^3}{4} \rho d\emptyset dz = \frac{10}{4} \times \rho^4 \times 2\pi \times 10$$

$$= 50\pi\, \rho^4 = 50\pi(2^4 - 1^4) = 750\pi$$

The right side is:

$$\int_{vol} (\nabla . \mathrm{D}) dv = \int_{vol} \left(\frac{1}{\rho}\frac{\partial}{\partial \rho} \rho D_\rho\right) dv = \int_{vol} \left(\frac{1}{\rho}\frac{\partial}{\partial \rho} \rho \left(\frac{10\rho^3}{4}\right)\right) dv$$

$$= \int_{vol} \left(\frac{1}{\rho}\frac{\partial}{\partial \rho} \left(\frac{10\rho^4}{4}\right)\right) dv = \int_{vol} (10\rho^2) dv = \int_0^{10} \int_0^{2\pi} \int_1^2 10\rho^2 \, \rho \, d\rho \, d\emptyset \, dz$$

$$= 10 \times 2\pi \times 10 \times \left[\frac{\rho^4}{4}\right]_1^2 = 750\pi$$

Example: Given that $\overrightarrow{\mathbf{D}} = 10\sin\theta\,\mathbf{a}_{\boldsymbol{r}} + 2\cos\theta\,\mathbf{a}_{\boldsymbol{\theta}}$, evaluate both sides of the divergence theorem for the volume enclosed by the shell r = 2?

Solution:

The left side of divergence theorem is:

$$\oint D_S\ .dS = \oint (10\sin\theta\,\mathbf{a}_{\boldsymbol{r}} + 2\cos\theta\,\mathbf{a}_{\boldsymbol{\theta}})\,.\,r^2\sin\theta\ \,d\theta d\emptyset\,\mathbf{a}_{\boldsymbol{r}}$$

$$\oint D_S\ .dS = \int_0^{2\pi}\int_0^{\pi} 10r^2\sin^2\theta\ \,d\theta d\emptyset = 10(2)^2\times 2\pi\times\int_0^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= 80\pi\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi} = 80\pi\left[\frac{1}{2}\pi\right] = 40\pi^2$$

The right side is:

$$\int_{vol}(\nabla.\mathrm{D})dv = \int_{vol}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2 D_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,D_\theta)\right)dv$$

$$= \int_{vol}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2(10\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,2\cos\theta)\right)dv$$

$$= \int_{vol}\left(\frac{1}{r^2}\frac{\partial}{\partial r}10r^2\sin\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin 2\theta\right)dv$$

$$= \int_{vol}\left(\frac{20}{r}\sin\theta + \frac{2}{r\sin\theta}\cos 2\theta\right)dv$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2}\left(\frac{20}{r}\sin\theta + \frac{2}{r\sin\theta}\cos 2\theta\right)r^2\sin\theta\ \,dr d\theta d\emptyset$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} 20r\sin^2\theta\ \,dr d\theta d\emptyset + \int_0^{2\pi}\int_0^{\pi}\int_0^{2} 2r\cos 2\theta\ \,dr d\theta d\emptyset$$

$$= [10r^2]_0^2\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi}[\emptyset]_0^{2\pi} + \ [r^2]_0^2\left[\frac{1}{2}\sin 2\theta\right]_0^{\pi}[\emptyset]_0^{2\pi}$$

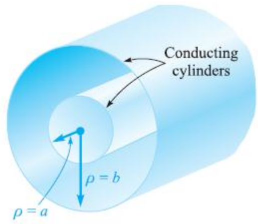
$$[10\times 4]\left[\frac{1}{2}\pi\right][2\pi] = 40\pi^2$$

Home work

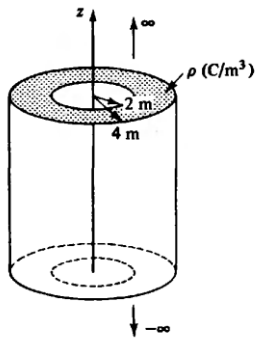
Q₁: In cylindrical coordinates, let $\rho_v = 0\ for\ \rho < 1\ mm$, $\rho_v = \sin(2000\pi\rho)\ nC/m^3\ for\ 1 \leq \rho < 1.5\ mm\ and\ \rho_v = 0\ for\ \rho > 1.5\ mm$. Find **D** everywhere?

Q₂: Spherical surfaces at r = 2, 4, and 6 m carry uniform surface charge densities of 20 nC/m^2, −4 nC/m^2, and ρ_{s0} respectively. (*a*) Find **D** at r = 1,3, and 5 m. (*b*) Determine ρ_{s0} such that **D** = 0 at r = 7 m.

Q₃: Let a 50-cm length of coaxial cable having an inner radius of 1 mm and an outer radius of 4 mm. The total charge on the inner conductor is 30 nC. Find the charge density on each conductor, and the E and D fields?



Q₄: *The volume in cylindrical coordinates between ρ= 2 m and ρ= 4 m contains a uniform charge density p (C/m^3). Use Gauss' law to find D in all regions*



Q₅: Given the flux density $D = \frac{16}{r}\cos(2\theta)\ C/m^2$, use two different methods to find the total charge within the region 1 < r < 2 m, 1 < θ< 2 rad, 1 < $\emptyset$< 2 rad.

Q₆: A point charge Q is at the origin, show that $\nabla . D = 0$ everywhere?

Q₇: Show that $\nabla . E = 0$ for the field of infinite sheet charge?